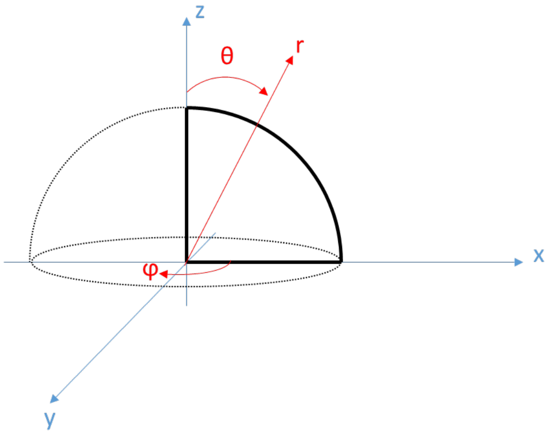
Exercises Module 4

Corrections

Exercise 4.1



In this problem we expect the concentration c_A to be a function of both r and θ, but not ϕ as the geometry is rotationally symmetric around ϕ. Therefore, we seek $c_A(r, \theta)$ over the domain of the hemisphere.

The differential equation used will be Eq. 4.16 since we have no reaction and the continuum is solid. Therefore:

$$\frac{\partial c_A}{\partial t} = \mathcal{D}_{AB} \nabla^2 c_A$$

In spherical coordinates :

$$\frac{\partial c_A}{\partial t} = \mathcal{D}_{AB} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_A}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial c_A}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 c_A}{\partial \phi^2} \right]$$

Since $c_A(r, \theta) \neq f(\phi)$ we can simplify:

$$\frac{\partial c_A}{\partial t} = \mathcal{D}_{AB} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c_A}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial c_A}{\partial \theta} \right) \right]$$

To solve this equation, we need 4 boundary conditions (2 for r and 2 for θ), and one initial condition.

Initial condition: for $t = 0$ $c_A = 0$ over the entire domain of the hemisphere

boundary conditions for t > 0:

at $r = R$, $c_A = c_{As}$ (surface of hemisphere)

at $\theta = \frac{\pi}{2}$, $c_A = c_{As}$ (bottom surface)

at $r = 0$, $c_A = c_{As}$ (this is single point at the origin, which is on the surface)

The last boundary condition is a bit tricky. We have to look at the symmetry in the system. We have an axis of symmetry when $\theta = 0$. The flux of A in the direction perpendicular to this direction must be 0 at this axis, due to the symmetry (continuity) condition. Since at $\theta = 0$, the theta direction, $\hat{\boldsymbol{\theta}}$ (or $\overrightarrow{e_\theta}$) is indeed perpendicular to the axis of symmetry, which is exactly what we need. Then we need only to write:

$$j_{A,\theta}\Big|_{\theta=0} = 0$$

Applying Fick's law of diffusion in 3D and taking only the theta component:

$$-\mathcal{D}_{AB}\left(\frac{1}{r}\frac{\partial c_A}{\partial \theta}\right)\Big|_{\theta=0} = 0$$

This simplifies to:

$$\frac{\partial c_A}{\partial \theta}\Big|_{\theta=0} = 0$$

Note: at $\theta = 0$, $\boldsymbol{j_A} \neq 0$. This is because the r-component of the flux of A will not equal zero (i.e. $j_{A,r}\Big|_{\theta=0} \neq 0$, or rather $\frac{\partial c_A}{\partial r}\Big|_{\theta=0} \neq 0$).

Exercise 4.2

The system is isotropic in θ and ϕ, and therefore the temperature will only spatially depend on the radius r:

$T = T(r,t)$

There are two different continuous media in the system through which the temperature will be transported: the bullet and the gel. Therefore, we need two differential equations to describe it. Starting with equation 4.8, applying spherical coordinates, and simplifying for $T = T(r,t)$ we have:

Inside the bullet:

$$\frac{\partial T_{bul}}{\partial t} = \frac{\alpha_{bul}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{bul}}{\partial r}\right)$$

Inside the gel:

$$\frac{\partial T_{gel}}{\partial t} = \frac{\alpha_{gel}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{gel}}{\partial r}\right)$$

The initial conditions are:

$$T_{bul}(r \leq R, t < 0) = T_i$$

$$T_{gel}(r \geq R, t < 0) = T_\infty$$

We need four boundary conditions (two for each equation):

$$T_{gel}\,(r \to \infty\,, t) = T_\infty$$

$$T_{bul}\,(r = R, t) = T_{gel}(r = R, t)$$

$$q_{bul}(r = R, t) = q_{gel}\,(r = R, t) \Leftrightarrow k_{bul}\left.\frac{\partial T_{bul}}{\partial r}\right|_{r=R,t} = k_{gel}\left.\frac{\partial T_{gel}}{\partial r}\right|_{r=R,t}$$

$$q_{bul}\,(r = 0, t) = 0 \;\Leftrightarrow \left.\frac{\partial T_{bul}}{\partial r}\right|_{r=0,t} = 0$$

Exercise 4.3

Under the stokes flow approximation we can assume that flow is only in the θ direction. Then $v_r = v_z = 0$ and we seek $v_\theta(r, z)$.

Flow is shear driven and pressure and gravity are ignored. Our simplified momentum balance in cylindrical coordinates becomes:

$$0 = \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\,(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right)$$

The boundary conditions are (*choosing the origin of the coordinate system at the center of the disk's bottom, at the interface with the fluid, and the positive z direction toward the ceramic shaft*):

$$v_\theta(r, z = 0) = \Omega r$$

$$v_\theta(r, z = \zeta) = 0$$

$$v_\theta(r = 0, z) = 0$$

$$\tau_{r\theta}(r = R, z) = 0 \;\rightarrow\; -\mu\left.\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)\right|_{r=R} = 0 \rightarrow \left.\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right|_{r=R} = 0 \qquad \text{(free surface)}$$

To solve this we need to make a reasonable guess for $v_\theta(r, z)$. Given the first B.C: above it is reasonable to guess $v_\theta(r, z) = r\,f(z)$.

Plugging this into the differential balance and simplifying (few steps omitted) we get:

$$0 = \frac{d^2 f}{dz^2}$$

Simplifying the B.C.s we get

$$f(z = 0) = \Omega$$

$$f(z = \zeta) = 0$$

Integrating the differential balance and applying the B.C.s, and plugging back into v_θ gives:

$$v_\theta(r, z) = \Omega r \left(1 - \frac{z}{\zeta}\right)$$

b) the torque is given by:

$$T_z = \int_A \tau_{z\theta}|_{z=0} r \, dA$$

Plugging in for dA and the stress we get

$$T_z = \int_0^{2\pi} \int_0^R \left(-\mu \frac{\partial v_\theta}{\partial z}\right)\Big|_{z=0} r^2 \, dr \, d\theta$$

And then

$$T_z = \frac{2\pi\mu\Omega}{\zeta} \int_0^R r^3 \, dr$$

Finally giving:

$$T_z = \frac{\pi\mu\Omega \mathrm{R}^4}{2\zeta}$$

(c) We want to find $Q_z = \int_A q_z|_{z=0} \, dA$, thus first we look for $T(r, z)$ in the liquid. Starting with eq. 4.6:

$$\rho c_p \frac{DT}{Dt} = (k\nabla^2 T) + \mu\Phi_v$$

Steady state:

$$\rho c_p(\boldsymbol{v} \cdot \boldsymbol{\nabla} T) = (k\nabla^2 T) + \mu\Phi_v$$

Expand in cylindrical coordinates:

$$\rho c_p \left(\begin{bmatrix} v_r & v_\theta & v_z \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial T}{\partial r} & \frac{1}{r}\frac{\partial T}{\partial \theta} & \frac{\partial T}{\partial z} \end{bmatrix} \right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right) + \mu\Phi_v$$

Considering that $T \neq f(\theta)$ and that $v_r = v_z = 0$ we have

$$0 = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right) + \mu\Phi_v$$

Further ignoring conduction in the *r* direction and expanding the term Φ_v :

$$0 = k\left(\frac{\partial^2 T}{\partial z^2}\right) + \mu\left[\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)^2 + \left(\frac{\partial v_\theta}{\partial z}\right)^2\right]$$

Substituting $v_\theta(r,z) = \Omega r\left(1 - \frac{z}{\zeta}\right)$ we get:

$$\frac{\partial^2 T}{\partial z^2} = -\frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2$$

The B.C.s are :

$$T(r, z = 0) = \mathrm{T}_1$$

$$\left.\frac{\partial T}{\partial z}\right|_{z=\zeta} = 0$$

Integrating the differential equation once gives:

$$\frac{\partial T}{\partial z} = -\frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2 z + g(r)$$

Here we can apply the second B.C. :

$$0 = -\frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2 \zeta + g(r)$$

Thus:

$$g(r) = \frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2 \zeta$$

Or rather:

$$\frac{\partial T}{\partial z} = -\frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2 (\zeta - z)$$

Integrating again:

$$T = \frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2 \left(\zeta z - \frac{z^2}{2}\right) + h(r)$$

Apply first B.C.:

$$T_1 = h(r)$$

Then we have our $T(r, z)$

$$T = \frac{\mu}{k}\left(\frac{r\Omega}{\zeta}\right)^2\left(\zeta z - \frac{z^2}{2}\right) + T_1$$

Now we can derive $q_z|_{z=0}$

$$q_z|_{z=0} = -k\frac{\partial T}{\partial z}\bigg|_{z=0} = -\frac{\mu r^2\Omega^2}{\zeta}$$

And finally

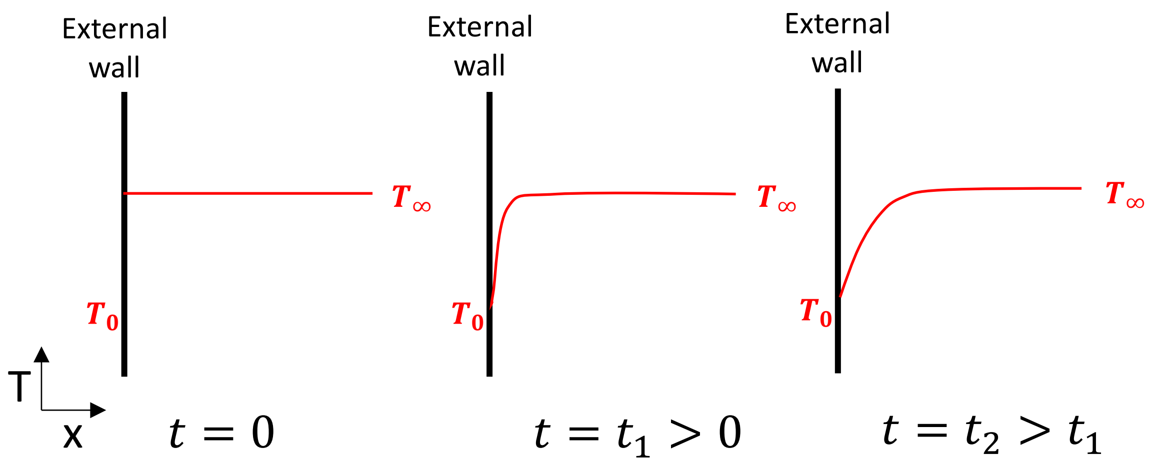
$$Q_z = \int_A q_z|_{z=0}\, dA = \int_0^{2\pi}\int_0^R -\frac{\mu r^2\Omega^2}{\zeta} r\, dr\, d\theta$$

$$Q_z = -\frac{2\pi\mu\Omega^2}{\zeta}\int_0^R r^3\, dr\, d\theta$$

$$Q_z = -\frac{\pi\mu\Omega^2 \mathrm{R}^4}{2\zeta}$$

Exercise 4.4

We consider the wall as semi-infinite space, which surface is suddenly exposed to a drop in temperature from T_∞= 40°C to $T_0 = 20°C$. The heat profile will evolve over time with the following trend:



Since the heat propagates inside a solid, we can neglect the convection in front of the conduction and therefore the heat differential equation becomes:

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}}$$

The equation needs to be non-dimensionalized. We are dealing with the same equation than in section 4.7 from the lecture, where the temperature replaces the speed and the thermal diffusivity replaces the kinematic viscosity. The solving of the equation is therefore identical as long as we can convert it to the same non-dimensional form.

We use the following non-dimensional variables:

$$\boldsymbol{\Theta} = \frac{\mathbf{T - T_\infty}}{\boldsymbol{T_0 - T_\infty}} \quad et \quad \boldsymbol{\eta} = \frac{\boldsymbol{x}}{\sqrt{\boldsymbol{4\alpha t}}}$$

Remark : In example in lecture, we had $v_\infty = 0$ so it did not appear in the non-dimensionalization

The boundary conditions are:

$$x = 0, \qquad T = T_0 \qquad \Rightarrow \qquad \boldsymbol{\eta = 0}, \quad \boldsymbol{\Theta = 1}$$

$$x = \infty, \qquad T = T_\infty \qquad \Rightarrow \qquad \boldsymbol{\eta = \infty}, \quad \boldsymbol{\Theta = 0}$$

Changing variables in the equation:

$$\frac{\boldsymbol{\partial \Theta}}{\boldsymbol{\partial t}} = \boldsymbol{\alpha} \frac{\boldsymbol{\partial^2 \Theta}}{\boldsymbol{\partial x^2}}$$

Note that this is the same differential equation as we found for the velocity profile as in lecture, for any exam you can just apply the solution we know already… however here is the solution again (method of combination of variables):

Then:

$$\frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial \eta} * \frac{\partial \eta}{\partial t} = \frac{x}{\sqrt{4\alpha}} * \left(-\frac{1}{2} t^{-\frac{3}{2}}\right) \frac{\partial \Theta}{\partial \eta} = -\frac{1}{2} \frac{\eta}{t} \frac{\partial \Theta}{\partial \eta}$$

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial \eta} * \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}} \frac{\partial \Theta}{\partial \eta}$$

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \Theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{4\alpha t}} \frac{\partial \Theta}{\partial \eta}\right) = \frac{\partial}{\partial \eta}\left(\frac{1}{\sqrt{4\alpha t}} \frac{\partial \Theta}{\partial \eta}\right) * \frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t} * \frac{\partial^2 \Theta}{\partial \eta^2}$$

The new equation is:

$$-\frac{1}{2} \frac{\eta}{t} \frac{\partial \Theta}{\partial \eta} = \frac{\alpha}{4\alpha t} \frac{\partial^2 \Theta}{\partial \eta^2}$$

$$\boxed{\frac{\boldsymbol{\partial^2 \Theta}}{\boldsymbol{\partial \eta^2}} + \boldsymbol{2\eta} \frac{\boldsymbol{\partial \Theta}}{\boldsymbol{\partial \eta}} = \boldsymbol{0}}$$

Let's set $\Gamma = \frac{\partial \Theta}{\partial \eta}$.

$$\frac{\partial \Gamma}{\partial \eta} + 2\eta\Gamma = 0$$

$$\frac{\partial \Gamma}{\Gamma} = -2\eta\,\partial\eta$$

$$\ln(\Gamma) = -\eta^2$$

$$\boldsymbol{\Gamma = C_1 e^{-\eta^2}}$$

This yields:

$$\frac{\partial \Theta}{\partial \eta} = C_1 e^{-\eta^2}$$

Integrating Θ between 0 and η :

$$\Theta(\eta) = C_1 \int_0^{\eta} e^{-\acute{\eta}^2} d\acute{\eta} + \Theta(0)$$

$$\Theta(\eta) = 1 + C_1 \int_0^{\eta} e^{-\acute{\eta}^2} d\acute{\eta}$$

On the other hand $\Theta(\infty) = 0$, therefore:

$$0 = 1 + C_1 \int_0^{\infty} e^{-\acute{\eta}^2} d\acute{\eta}$$

$$C_1 = -\frac{2}{\sqrt{\pi}}$$

And therefore:

$$\Theta(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\acute{\eta}^2} d\acute{\eta}$$

$$\boxed{\boldsymbol{\Theta(\eta) = \mathrm{erfc}\,\eta}}$$

i.e.

$$\boxed{\boldsymbol{\frac{T - T_\infty}{T_0 - T_\infty} = \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)}}$$

For $x_1 = 0.05\,m$ and $t_1 = 900s$,

$$\eta_1(x_1, t_1) = 1.18$$

Therefore

$$\mathrm{erfc}(\eta_1) = 0.095 \text{ (tabulated)}$$

Finally:

$$T(x_1, t_1) = (T_0 - T_\infty)\,\mathrm{erfc}(\eta_1) + T_\infty = -20 * 0.095 + 40$$

$$\underline{\boldsymbol{T(x_1, t_1) = 38.1°C}}$$

The semi-infinite approximation is satisfying as long as the Fo < 0.2 at the center of the wall.

Meaning that,

$$Fo = \frac{\alpha t}{Y^2} \leq 0.2$$

then

$$\boldsymbol{t} \leq \frac{Y^2 * 0.2}{\boldsymbol{\alpha}} = \frac{(1m)^2 * 0.2}{\mathbf{5 \cdot 10^{-7}} \boldsymbol{m^2/s}} = \mathbf{4 * 10^5} \boldsymbol{s}$$

The approximation is valid for any time lower than 4.6 days (this is a long time because α is very small and therefore the heat propagates very slowly inside the wall)

Exercise 4.5

The fluid is moving in the x direction and since $L, b \gg h$, the speed will vary only in the y direction:

$$v = v_x(t, y)$$

And if we neglect the effects of gravity: $p = p_{atm}$ in all the system.

The NS equation projected on x gives:

$$0 = \rho \frac{\partial v_x}{\partial t} - \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\boxed{\boldsymbol{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}}}$$

Since 100 s is relatively short, we can start by making the semi-infinite approximation (to be checked later).

Non-dimensionalization and solution (same as in lecture, for any exam you can just apply the solution we know already… but just to see it again):

We set:

$$\begin{cases} \phi = \dfrac{v_x}{v_0} \\ \eta = \dfrac{y}{\sqrt{4\nu t}} \end{cases}$$

Then:

$$\frac{\partial v_x}{\partial t} = v_0 \frac{\partial \phi}{\partial t}$$

$$\frac{\partial^2 v_x}{\partial y^2} = v_0 \frac{\partial^2 \phi}{\partial y^2}$$

Next:

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{d\eta} * \frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\frac{d\phi}{d\eta}$$

$$\frac{\partial \phi}{\partial y} = \frac{d\phi}{d\eta} * \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}}\frac{d\phi}{d\eta} \Longrightarrow \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\sqrt{4\nu t}}\frac{d^2\phi}{d\eta^2} * \frac{\partial \eta}{\partial y} = \frac{1}{4\nu t}\frac{d^2\phi}{d\eta^2}$$

Therefore, the non-dimensional differential equation is:

$$-\frac{\eta}{2t}\frac{d\phi}{d\eta} = \nu * \frac{1}{4\nu t}\frac{d^2\phi}{d\eta^2}$$

$$\boxed{\boldsymbol{\frac{d^2\phi}{d\eta^2} + 2\eta\frac{d\phi}{d\eta} = 0}}$$

Integration of the differential equation

We set

$$\psi = \frac{d\phi}{d\eta}$$

The equation becomes

$$\frac{d\psi}{d\eta} + 2\eta\psi = 0$$

$$\frac{d\psi}{\psi} = -2\eta\, d\eta$$

$$\psi(\eta) = C_1 e^{-\eta^2}$$

i.e.

$$\frac{d\phi}{d\eta} = C_1 e^{-\eta^2}$$

Therefore:

$$\phi\ (\eta) = C_1 \int_0^{\eta} e^{-\eta'^2} d\eta' + C_2$$

The boundary conditions are:

$$(\mathrm{B.C.}\,1)\ v_x(0, t) = v_0 \Leftrightarrow \ \phi\ (0) = 1$$

$$(\mathrm{B.C.}\,2)\ v_x(\infty, t) = 0 \Leftrightarrow \phi(\infty) = 0$$

$$\mathrm{B.C.}\,1\ \Rightarrow C_2 = 1$$

$$\mathrm{B.C.}\,2\ \Rightarrow 1 + \ C_1 \int_0^{\infty} e^{-\eta'^2} d\eta' = 0\ \Rightarrow \mathrm{C}_1 = -\frac{2}{\sqrt{\pi}}$$

Finally

$$\boxed{\boldsymbol{\phi\ (\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta'^2} d\eta' = \mathbf{erfc}\ (\eta)}}$$

The overall solution, using the semi-infinite model is therefore:

$$\boxed{\boldsymbol{v_x(y, t) = v_0 * \mathbf{erfc}\left(\frac{y}{\sqrt{4\nu t}}\right)}}$$

a) $y = 1\ cm$ and $t = 100s$

Then

$$\eta = \frac{0.01}{\sqrt{4 * 10^{-6} * 100}} = 0.5 \Rightarrow \ \mathrm{erfc}\ (\eta) = 0.4795$$

Therefore

$$v_x\ (1\ cm, 100s) = 0.1 * 0.4795 = \boldsymbol{0.048\ m.s^{-1}}$$

b) We consider the fluid stagnant when

$$\phi = 0.001\ (\mathrm{arbitrary})$$

$$0.001 = \mathrm{erfc}\ (\eta)\ \ \Rightarrow\ \eta \approx 2.35$$

$$\frac{y}{\sqrt{4\nu t}} = 2.35\ \Leftrightarrow y = 2.35 * \sqrt{4 * 10^{-6} * 100} = \boldsymbol{4.70\ cm}$$

c) we can consider that the semi-infinite approximation is valid for small Fo numbers:

$$Fo = \frac{\nu t}{h^2} = 0.2 \qquad \text{(arbitrary, 0.1 would be better)}$$

$$\frac{0.2\,(0.06)^2}{10^{-6}} = \mathrm{t} \implies t = 720\ sec$$

The semi-infinite approximation is reasonable for the first 720 s after the plate is put in motion.

d) The shear stress on the plate is homogenous over the plate, and therefore:

$$F_T = Lb * \left(-\mu \frac{\partial v_x}{\partial y}\bigg|_{y=0} \right)$$

$$F_T = -\mu L b v_0 \frac{\partial}{\partial y}[\mathrm{erfc}(\eta)]\bigg|_{y=0}$$

$$F_T = -\mu L b v_0 \left[\frac{\partial}{\partial \eta}[\mathrm{erfc}(\eta)] * \frac{\partial \eta}{\partial y}\right]_{y=0}$$

$$F_T = -\mu L b v_0 \left[-\frac{2}{\sqrt{\pi}} e^{-\eta^2} * \frac{1}{\sqrt{4\nu t}}\right]_{y=0}$$

$$\boxed{\boldsymbol{F_T(t) = \frac{\mu L b v_0}{\sqrt{\pi \nu t}} = \rho L b v_0 \sqrt{\frac{\nu}{\pi t}}}}$$

Therefore, at t=163s

$$\boldsymbol{F_T(720\ s) = 10^3 * 0.1 * Lb \sqrt{\frac{10^{-6}}{\pi * 720}} = 0.002\ Lb\ [N]}$$